(3) The given eqⁿ is $\frac{d^2y}{dx^2} + y = \text{cosec}x.$

A.E. is $D^2+1=0.$

$\therefore \quad D = \pm i$

C.F. $= e^{0x}(c_1\cos x + c_2 \sin x)$

$\Rightarrow c_1\cos x + c_2\sin x$

Now P.I $= \frac{1}{D^2+1}\text{cosec}x \Rightarrow \frac{1}{(D+i)(D-i)}\text{cosec}x$

$\Rightarrow \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right]\text{cosec}x$

{Resolving into Partial fraction}.

$\Rightarrow \frac{1}{2i}\left[\frac{1}{D-i}\text{cosec}x - \frac{1}{D+i}\text{cosec}x\right].$

$\Rightarrow \frac{1}{2i}\left[e^{ix}\int\frac{e^{-ix}}{\sin x}dx - e^{-ix}\int\frac{e^{ix}}{\sin x}dx\right].$

$\left\{\because \frac{1}{D-a}X = e^{ax}\int e^{-ax}\cdot X\,dx\right\}.$

$\Rightarrow \frac{1}{2i}e^{ix}\int\frac{\cos x - i\sin x}{\sin x}dx - \frac{1}{2i}e^{-ix}\int\frac{\cos x + i\sin x}{\sin x}dx$

$\{\because e^{\pm i\theta} = \cos\theta \pm i\sin\theta\}.$

$\Rightarrow \frac{1}{2i}e^{ix}\int(\cot x - i)\,dx - \frac{1}{2i}e^{-ix}\int(\cot x + i)\,dx$

$\Rightarrow \frac{1}{2i}e^{ix}[\log\sin x - ix] - \frac{e^{-ix}}{2i}[\log\sin x + ix].$

$\Rightarrow \frac{1}{2i}\log\sin x\,(e^{ix} - e^{-ix}) - \frac{x}{2}[e^{ix} + e^{-ix}].$

$\Rightarrow$ $\log \sin x \left[\frac{e^{ix} - e^{-ix}}{2i} \right] - x \left[\frac{e^{ix} + e^{-ix}}{2} \right].$

$\Rightarrow$ $\log \sin x \cdot \sin x - x \cos x$

Hence, complete solution is C.F. + P.I

$y = c_1 \cos x + c_2 \sin x + \sin x . \log \sin x - x \cos x.$

(4) $$\frac{d^2y}{dx^2} + y = \sec x.$$

The given equation is

$$\frac{d^2y}{dx^2} + y = \sec x$$

Symbolic form

$$(D^2+1)\,y = \sec x$$

A.E. is

$$D^2+1=0 \Rightarrow D^2=-1 \Rightarrow D = \pm i$$

$$\text{C.F.} = e^{0x}(c_1\cos x + c_2 \sin x)$$

$$= c_1 \cos x + c_2 \sin x.$$

Now. $$\text{P.I.} = \frac{1}{D^2+1}\sec x$$

$$= \frac{1}{(D+i)(D-i)}\sec x \Rightarrow \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right]\sec x$$

(Resolving into partial fraction).

$$\Rightarrow \frac{1}{2i}\left[\frac{1}{D-i}\sec x - \frac{1}{D+i}\sec x\right].$$

$$\Rightarrow \frac{1}{2i}\left[e^{ix}\int \frac{e^{-ix}}{\cos x}dx - e^{-ix}\int \frac{e^{ix}}{\cos x}dx\right]$$

$$\Rightarrow \frac{1}{2i}e^{ix}\int \frac{\cos x - i\sin x}{\cos x}dx - \frac{1}{2i}e^{-ix}\int \frac{\cos x + i\sin x}{\cos x}dx$$

$$\{\because e^{\pm i\theta} = \cos\theta \pm i\sin\theta\}.$$

$$\Rightarrow \frac{1}{2i}e^{ix}\int (1 - i\tan x)\,dx - \frac{e^{-ix}}{2i}\int (1+i\tan x)\,dx.$$

$\Rightarrow \frac{1}{2i} e^{ix}[x + i\log|\cos x|] - \frac{e^{-ix}}{2i}[x - i\log|\cos x|]$

$\Rightarrow \frac{1}{2i}[x(e^{ix} - e^{-ix})] + \frac{1}{2}\log\cos x(e^{ix} + e^{-ix})$

$\Rightarrow x\left(\frac{e^{ix} - e^{-ix}}{2}\right) + \log\cos x\left(\frac{e^{ix} + e^{-ix}}{2}\right).$

$\Rightarrow x\sin x + \log(\cos x)\cos x$

$\left\{\because \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \text{ and } \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}\right\}.$

$\Rightarrow x\sin x + \cos x \cdot \log\cos x$

Hence solution is C.F + P.I.

$y = c_1\cos x + c_2\sin x + x\sin x + \cos x . \log\cos x.$

(5) The given eq^n is

$$(D^2 - 3D + 2)y = \cos e^{-x}$$

A.E. is

$$D^2 - 3D + 2 = 0$$

$$(D-2)(D-1) = 0 \Rightarrow D = 1, 2$$

C.F. $c_1 e^{x} + c_2 e^{2x}$.

Now P.I. $\Rightarrow \frac{1}{D^2 - 3D + 2} \cos e^{-x}$

$\Rightarrow \frac{1}{(D-2)(D-1)} \cos e^{-x}$

$\Rightarrow \frac{1}{(D-2)}\left(\frac{1}{(D-1)} \cos e^{-x}\right)$

$\Rightarrow \frac{1}{(D-2)}\left(e^{x}\int e^{-x}\cos e^{-x}dx\right)$

$\left[\because \frac{1}{(D-a)}X \Rightarrow e^{ax}\int (e^{-ax}\cdot X)dx\right].$

$\Rightarrow \frac{1}{D-2}\left((-e^{x})\int \cos e^{-x}(-e^{-x})dx\right)$

$\Rightarrow \frac{1}{D-2}[(-e^{x})\sin e^{-x}].$

$\Rightarrow e^{2x}\int e^{-2x}(-e^{x}\sin e^{-x})dx$

$\Rightarrow e^{2x}\int \sin e^{-x}(-e^{-x})dx$

$\Rightarrow e^{2x}(-\cos e^{-x})$

$\Rightarrow -e^{2x}\cos e^{-x}$

complete solution is

$y = C.F + P.I$

$y = c_1e^{x} + c_2e^{2x} - e^{2x}\cos e^{-x}.$